Superposition in nonlinear wave and evolution equations

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Real and bounded elliptic solutions suitable for applying the Khare-Sukhatme superposition procedure are presented and used to generate superposition solutions of the generalized modified Kadomtsev-Petviashvili equation (gmKPE) and the nonlinear cubic-quintic Schrödinger equation (NLCQSE).

1 Introduction

As has been shown recently [Cooper *et al.* 2002], [Khare *et al.* 2002a], [Khare *et al.* 2003], [Khare *et al.* 2002b] (periodic) Jacobian elliptic functions (if they are solutions of a certain nonlinear wave and evolution equation (NLWEE)) are start solutions for generating new solutions of the NLWEE by a linear superposition procedure. Thus, elliptic functions are of specific importance for finding solutions of NLWEEs. On the other hand, based on a symmetry reduction method, a technique to obtain elliptic solutions of certain NLWEEs was proposed and applied to the gmKPE and the NLCQSE [Schürmann 1996], [Schürmann *et al.* 2004a], [Schürmann *et al.* 2004b].

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It is the aim of the present paper to combine these approaches in order to obtain general elliptic solutions that can serve as start solutions for superposition ("suitable solutions").

The superposition procedure can be described as follows [Cooper *et al.* 2002]: If a solution of a NLWEE$[\psi(x, y, t)] = 0$ can be expressed in terms of Jacobian elliptic functions

$$\Psi(x, y, t) = \sum_{\nu=0}^{l} a_\nu \text{qn}^\nu \left[\mu(x + ky + vt), m\right],\tag{1}$$

where qn is anyone of the Jacobian elliptic functions and a_ν, μ, c are constants, then the superposition solution [Cooper *et al.* 2002, Eqs. (4), (14)]

$$\widetilde{\Psi}(x, t) = \sum_{\lambda=1}^{p} \sum_{\nu=0}^{l} a_\nu \text{qn}^\nu \left[\mu(x + ky + v_p t) + \frac{n(\lambda - 1)K(m)}{p}, m\right],\tag{2}$$

where $n \in \{2, 4\}$ (depending on the periodicity of the Jacobian elliptic function and on ν) and $K(m), m$ denote the complete elliptic integral of first kind and the modulus parameter ($0 \leq m \leq 1$), respectively, also may be a solution of the NLWEE. The number p is integer (it depends on the NLWEE whether it is even or/ and odd) and the speed v_p can be determined by using certain remarkable, recently established, identities involving Jacobian elliptic functions [Khare *et al.* 2002a], [Khare *et al.* 2003]. It should be noted, that the existence of solutions (1) of a certain NLWEE does not necessarily imply the existence of a solution (2). As shown in Refs. [Cooper *et al.* 2002], [Khare *et al.* 2002b] solutions (2) exist for the Korteweg-de Vries equation (KdV), the Kadomtsev-Petviashvili equation (KP), the nonlinear (cubic) Schrödinger equation (NLSE), the $\lambda \phi^4-$field equation, the Sine-Gordon equation and the Boussinesq equation. On the other hand, it may happen, as will be seen below, that a solution (1) is known but does not lead to a solution (2). It is crucial for the procedure, that appropriate relations between

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Jacobian elliptic functions are known.

The symmetry reduction approach can be outlined as follows [Schürmann *et al.* 2004b]: The NLWEE[$\psi(x,t)$] $= 0$ is reduced by an appropriate transformation $\psi \rightarrow f$ (e. g., $\psi(x,t) = f(z), z = x - ct$), where f is supposed to obey the ordinary nonlinear differential equation ("basic equation")

$$\left(\frac{df(z)}{dz}\right)^2 = \alpha f^4 + 4\beta f^3 + 6\gamma f^2 + 4\delta f + \epsilon \equiv R(f), \tag{3}$$

(with real z, $f(z)$, α, β, γ, δ, ϵ), leading to an equation $P(f) = 0$, where P denotes a polynomial in f. Vanishing coefficients in the polynomial equation $P(f) = 0$ imply equations which partly determine the coefficients α, β, γ, δ, ϵ in Eq. (3). In general, the coefficients depend on the structure and parameters of the NLWEE and, finally, on the parameters of the transformation $\psi \rightarrow f$. Thus, the problem of finding a solution of the NLWEE is reduced to finding an appropriate transformation that leads to the basic equation (3).

As is well known [Weierstrass 1915, Whittaker *et al.* 1927] the solution $f(z)$ of Eq. (3) can be written as

$$f(z) = f_0 + \frac{R'(f_0)}{4[\wp(z; g_2, g_3) - \frac{1}{24}R''(f_0)]}, \tag{4}$$

where f_0 is a simple root of $R(f)$ [1] and the prime denotes differentiation with respect to f.

The invariants g_2, g_3 of Weierstrass' elliptic function $\wp(z; g_2, g_3)$ are related to the coefficients of $R(f)$ by [Chandrasekharan 1985]

$$g_2 = \alpha\epsilon - 4\beta\delta + 3\gamma^2, \tag{5}$$

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$$g_3 = \alpha\gamma\epsilon + 2\beta\gamma\delta - \alpha\delta^2 - \gamma^3 - \epsilon\beta^2. \tag{6}$$

The discriminant (of $\wp$ and R [Chandrasekharan 1985])

$$\triangle = g_2^3 - 27g_3^2, \tag{7}$$

is suitable to classify the behaviour of $f(z)$. The conditions

$$\triangle \neq 0 \quad \text{or} \quad \triangle = 0, \quad g_2 > 0, \quad g_3 > 0. \tag{8}$$

lead to periodic solutions [Schürmann *et al.* 2004b], whereas the conditions [Abramowitz *et al.* 1972]

$$\triangle = 0, \quad g_2 \geq 0, \quad g_3 \leq 0 \tag{9}$$

are associated with solitary wave like solutions.

Physical solutions $f(z)$ must be real and bounded. Considering the phase diagram of $R(f)$ [Schürmann 1996], [Drazin 1983] one obtains conditions, expressed in terms of the coefficients of the basic equation, that determine physical solutions. For convenience these conditions are referred to as the phase diagram conditions (PDC) in the following.

2   Elliptic start solutions for superposition

To apply the superposition procedure it is important to know whether a solution of the NLWEE according to (1) exists. To check this it is useful to rewrite Weierstrass' function $\wp$ as [2]

$$\wp(z) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m)}, \tag{10}$$

where $e_1 \geq e_2 \geq e_3$ denote the roots of the equation

$$4s^3 - g_2 s - g_3 = 0. \tag{11}$$

Substitution of Eq. (10) into Eq. (4) yields [3]

$$f(z) = \frac{(\alpha f_0{}^3 + 4\beta f_0{}^2 + 2e_3 f_0 + 5\gamma f_0 + 2\delta)\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m) + 2(e_1 - e_3)f_0}{(-\alpha f_0{}^2 - 2\beta f_0 + 2e_3 - \gamma)\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m) + 2(e_1 - e_3)}, \tag{12}$$

with $m = \frac{e_2 - e_3}{e_1 - e_3}$. Comparison with Eq. (1) shows that

$$-\alpha f_0{}^2 - 2\beta f_0 + 2e_3 - \gamma = 0 \tag{13}$$

is a necessary and sufficient condition that defines the subset of solutions (1). If $\alpha = 0$ holds the simple root f_0 of $R(f)$ can be choosen such that Eq. (13) and PDC are satisfied [4]. If $\alpha \neq 0$ and $\beta = \delta = 0$ Eq. (13) is satisfied also. If $\alpha \neq 0$ and $\beta \neq 0$, $\delta = \epsilon = 0$, Eq. (13), PDC, and the condition $\triangle = 0$, $g_3 > 0$ are not consistent, so that trigonometric functions (which are possible for $\triangle = 0$, $g_3 > 0$) are not suitable for superposition, because $f(z)$ is a constant according to the general solution of Eq. (3) [1].

Equation (13) represents a relation between the parameters $\{\alpha, \beta, \gamma, \delta, \epsilon\}$ and thus determines a subset of parameters of the problem modelled by the NLWEE for which further solutions can be generated by superposition according to Eq. (2). Combining Eqs. (12) and (13) (with $\alpha = 0$) we obtain

$$f(z) = \frac{2e_3 - \gamma}{2\beta} + \frac{12e_3^2 - 3\gamma^2 + 4\beta\delta}{4\beta(e_1 - e_3)}\mathrm{sn}^2(\sqrt{e_1 - e_3}z, m), \tag{14}$$

where e_1, e_3 must be chosen as the largest and smallest roots of Eq. (11), respectively, so that the condition (13) is valid for a simple root f_0 of Eq. (3) that satisfies the PDC.

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Equation (14) can be evaluated explicitely subject to the two cases $\alpha = 0$ and $\alpha \neq 0$, $\beta = \delta = 0$, respectively. If $\alpha = 0$ and, for simplicity, $\epsilon = 0$ the start solutions for superposition are

$$
f(z) = \begin{cases}
-\frac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{4\beta} \mathrm{dn}^2 \left(\frac{1}{2}\sqrt{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}z, \frac{2\sqrt{9\gamma^2 - 16\beta\delta}}{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}} \right), & \beta\delta > 0,\ \gamma > 0, \\[3ex]
\frac{4\delta}{-3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}} \mathrm{sn}^2 \left(\frac{1}{2}\sqrt{-3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}z, \frac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{3\gamma - \sqrt{9\gamma^2 - 16\beta\delta}} \right), & \beta\delta > 0,\ \gamma < 0, \\[3ex]
-\frac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{4\beta} \mathrm{cn}^2 \left(\frac{(9\gamma^2 - 16\beta\delta)^{\frac{1}{4}}}{\sqrt{2}}z, \frac{3\gamma + \sqrt{9\gamma^2 - 16\beta\delta}}{2\sqrt{9\gamma^2 - 16\beta\delta}} \right), & \beta\delta < 0,
\end{cases}
$$
(15)

where the various possibilities to satisfy (11) and (13) have been taken into account and $\triangle = 4\beta^2\delta^2(9\gamma^2 - 16\beta\delta) > 0$ is necessary and sufficient to fulfill PDC [Bronstein *et al.* 2000].

If $\alpha \neq 0$, $\beta = \delta = 0$ the start solutions read

$$
h(z) = \begin{cases}
-\frac{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}{\alpha} \mathrm{dn}^2 \left(\sqrt{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}z, \frac{2\sqrt{9\gamma^2 - \alpha\epsilon}}{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}} \right), & \alpha < 0,\ \gamma > 0,\ \epsilon < 0, \\[3ex]
\frac{\epsilon}{-3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}} \mathrm{sn}^2 \left(\sqrt{-3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}z, \frac{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}{3\gamma - \sqrt{9\gamma^2 - \alpha\epsilon}} \right), & \alpha > 0,\ \gamma < 0,\ \epsilon > 0, \\[3ex]
-\frac{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}{\alpha} \mathrm{cn}^2 \left(\sqrt{2}(9\gamma^2 - \alpha\epsilon)^{\frac{1}{4}}z, \frac{3\gamma + \sqrt{9\gamma^2 - \alpha\epsilon}}{2\sqrt{9\gamma^2 - \alpha\epsilon}} \right), & \alpha < 0,\ \epsilon > 0,
\end{cases}
$$
(16)

where $\triangle = 64\alpha^2\epsilon^2(9\gamma^2 - \alpha\epsilon) > 0$ and - according to the Cartesian sign rule - three numbers of sign reversals in the sequence of coefficients of $R(h)$ or $\triangle > 0$ and $\alpha > 0$ and two sign reversals to fulfill PDC.

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To sum up, Eqs. (15) and (16) represent all elliptic solutions with $\alpha = 0$, $\epsilon = 0$ and $\alpha \neq 0$, $\beta = \delta = 0$, respectively, that are suitable for the the procedure suggested by Khare and Sukhatme. "All elliptic" means that the solutions presented in Refs. [Cooper *et al.* 2002], [Khare *et al.* 2002b] are particular cases of Eqs. (15), (16). "Suitable" includes that the superposition procedure may fail if solutions according to Eq. (15) or (16) are inserted into the NLWEE in question leading to conditions that cannot be evaluated with respect to v_p (cf. Eq. (2)) because the associated relations between Jacobian functions are unknown (cf. Sec. 3). Examples to obtain superposition solutions are presented in the following. Equation (14) can be evaluated in the same manner subject to the PDC to obtain physical elliptic solutions if the simplifying assumption $\epsilon = 0$ does not hold.

3 Superposition solutions of the generalized modified Kadomtsev-Petviashvili equation

The approach outlined in the previous section can be elucidated by investigation of the gmKPE (Superposition solutions of the NLCQSE are presented in A.)

$$\psi_{xt} + \left((a + b\psi^q)\, \psi^q \psi_x \right)_x + c\psi_{xxxx} - \sigma^2 \psi_{yy} = 0, \tag{17}$$

where a, b, c, q, σ^2 are real constants. As shown previously [Schürmann *et al.* 2004a] elliptic traveling-wave solutions to Eq. (17) exist. The set of these solutions is determined by

$$
\begin{aligned}
\psi(x,y,t) &= f(z)^{1/q}, q \neq 0, \\
z &= x + ky + vt, \\
f_z^2 &= \alpha f^4 + 4\beta f^3 + 6\gamma f^2 + 4\delta f + \epsilon,
\end{aligned}
\tag{18}
$$

where α, β, γ, δ, ϵ are given by Eqs. (16a)-(16g) in Ref. [Schürmann *et al.* 2004a]. As shown above, the conditions $\alpha = 0$ or $\beta = \delta = 0$, $\alpha \neq 0$ lead to suitable solutions. Imposing additionally the PDC and condition (13), respectively, the parameters of solutions (18) are

$$q = \frac{1}{2}, \alpha = -\frac{b}{12c}, \beta = 0, \gamma = \frac{k^2\sigma^2 - v}{24c}, \delta = 0, \epsilon \neq 0, c \neq 0, \tag{19}$$

$$q = 1, \alpha = 0, \beta = -\frac{a}{12c}, \gamma = \frac{k^2\sigma^2 - v}{6c}, \epsilon = 0, c \neq 0, \tag{20}$$

$$q = 1, \alpha = -\frac{b}{6c}, \beta = 0, \gamma = \frac{k^2\sigma^2 - v}{6c}, \delta = 0, \epsilon \neq 0, c \neq 0, \tag{21}$$

$$q = 2, \alpha = 0, \beta = -\frac{a}{6c}, \gamma = \frac{2(k^2\sigma^2 - v)}{3c}, \epsilon = 0, c \neq 0.. \tag{22}$$

Referring to (19) and (20) first, solutions according to (16) and (15), respectively, have to be evaluated. Inserting (19) into (16), one obtains the suitable start solutions

$$\psi(x, y, t) = \begin{cases} B_1 \operatorname{dn}^2 \left[\mu_1(x + ky + vt), m_1 \right], \frac{b}{c} > 0, \ \frac{k^2\sigma^2 - v}{c} > 0, \ \epsilon < 0, \\ \\ B_2 \operatorname{sn}^2 \left[\mu_2(x + ky + vt), m_2 \right], \frac{b}{c} < 0, \ \frac{k^2\sigma^2 - v}{c} < 0, \ \epsilon > 0, \\ \\ B_3 \operatorname{cn}^2 \left[\mu_3(x + ky + vt), m_3 \right], \frac{b}{c} > 0, \ \epsilon > 0, \end{cases} \tag{23}$$

where B_j, μ_j, m_j are determined by inserting the parameters α, γ, ϵ according to (19) into (16). Formally the same result is obtained by inserting (20) into (15). Referring, secondly, to (21) the solutions follow from (16) as

$$\psi(x,y,t) = \begin{cases} B_1 \, \mathrm{dn}\left[\mu_1(x+ky+vt),m_1\right], \frac{b}{c} > 0, \, \frac{k^2\sigma^2-v}{c} > 0, \, \epsilon < 0, \\[12pt] B_2 \, \mathrm{sn}\left[\mu_2(x+ky+vt),m_2\right], \frac{b}{c} < 0, \, \frac{k^2\sigma^2-v}{c} < 0, \, \epsilon > 0, \\[12pt] B_3 \, \mathrm{cn}\left[\mu_3(x+ky+vt),m_3\right], \frac{b}{c} > 0, \, \epsilon > 0, \end{cases} \quad (24)$$

where (again) B_j, μ_j, m_j are determined from (16) with parameters according to (21). Formally the same results are given by (22) and (15).

According to Eq. (2) the first solution in (24) leads to a superposition solution for $p = 2$

$$\begin{aligned} \widetilde{\psi}(x,y,t) &= B \sum_{i=1}^{2} \mathrm{dn}\left(\mu(x+ky+v_2t)+(i-1)K(m),m\right), \\ B &= B_1, \mu = \mu_1, m = m_1. \end{aligned} \quad (25)$$

Inserting $\widetilde{\psi}(x,y,t)$ (denoting $\mathrm{d}_i = \mathrm{dn}\left(\mu(x+ky+v_2t)+(i-1)K(m),m\right)$) into Eq. (17) ($a = 0$, because $\beta = 0$ according to (21)) we get

$$\begin{aligned} &\left(-Bm\mu v_2 - Bc\mu^3(2m-m^2)\right)\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i + \sigma^2 Bm\mu k \frac{d}{dy}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i \\ &+2bB^3m^2\mu^2\sum_{i=1}^{2}\mathrm{d}_i\left(\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i\right)^2 - m\mu b B^3\left(\sum_{i=1}^{2}\mathrm{d}_i\right)^2\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i \quad (26) \\ &\qquad\qquad\qquad +6Bcm\mu^3\frac{d}{dx}\sum_{i=1}^{2}\mathrm{d}_i^2\mathrm{s}_i\mathrm{c}_i = 0. \end{aligned}$$

The last three terms of Eq. (26) can be simplified as follows.

Using $\mathrm{d}_1\mathrm{d}_2 = \sqrt{1-m}$ and $\mathrm{c}_1\mathrm{s}_1\mathrm{d}_2 + \mathrm{c}_2\mathrm{s}_2\mathrm{d}_1 = 0$ [Khare *et al.* 2002a, Eqs. (31), (39)] we obtain

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$$\left(\sum_{i=1}^{2}\mathrm{d}_i\right)^2\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i = \sum_{i=1}^{2}\mathrm{d}_i^2\mathrm{s}_i\mathrm{c}_i + \sqrt{1-m}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i. \tag{27}$$

Evaluating $\frac{d}{dx}\left(\left(\sum_{i=1}^{2}\mathrm{d}_i\right)^2\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i\right)$ and using Eq. (27), Eq. (26) can be rewritten as

$$\left(-Bm\mu v_2 - Bc\mu^3(2m-m^2) - m\mu b B^3\sqrt{1-m}\right)\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i \tag{28}$$

$$+\sigma^2 Bm\mu k\frac{d}{dy}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i + Bm\mu\left(6c\mu^2 - bB^2\right)\frac{d}{dx}\sum_{i=1}^{2}\mathrm{d}_i^2\mathrm{s}_i\mathrm{c}_i = 0.$$

The expression $(6c\mu^2 - bB^2)$ vanishes identically [5]. With $\frac{d}{dy}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i = k\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i$ Eq. (28) reads

$$\left(-Bm\mu v_2 - Bc\mu^3(2m-m^2) - m\mu b B^3\sqrt{1-m} + \sigma^2 Bm\mu k^2\right)\frac{d}{dx}\sum_{i=1}^{2}\mathrm{s}_i\mathrm{c}_i = 0, \tag{29}$$

so that the speed v_2 is given by

$$v_2 = -c\mu^2(2-m) - bB^2\sqrt{1-m} + \sigma^2 k^2. \tag{30}$$

Thus, we have found a superposition solution of Eq. (17) for this particular case. The start solution and the superposition solution are shown in Fig. 1.

We can generate superposition solutions for $p = 3$ from (23). As an example we consider the solution of the form dn^2 in detail and compare it with the results of Cooper, Khare and Sukhatme [Cooper *et al.* 2002]. According to Eq. (2) the superposition ansatz reads

$$\widetilde{\psi}(x,y,t) = B\sum_{i=1}^{3}\mathrm{dn}^2\left(\mu(x+ky+v_3t) + \frac{2(i-1)K(m)}{3}, m\right),$$
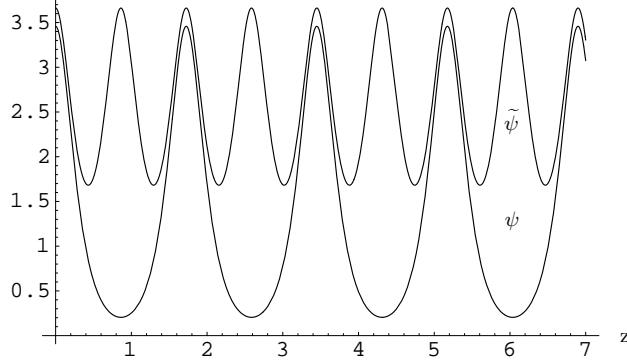
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Figure 1: The start solution $\psi(z)$ (cf. first solution of Eq. (24)) and the super-position solution $\widetilde{\psi}(z)$ (cf. Eq. (25)) for $\alpha = -2$, $\gamma = 4$, $\epsilon = -1$, $z = x + ky + vt$ and $z = x + ky + v_2 t$, respectively.

$$B = B_1, \mu = \mu_1, m = m_1. \tag{31}$$

Inserting $\widetilde{\psi}(x, y, t)$ (denoting $d_i = \mathrm{dn}\left(\mu(x + ky + v_3 t) + \frac{2(i-1)K(m)}{3}, m\right)$) into Eq. (17), we obtain

$$2Bm\mu(v_3 + 8c\mu^2 - 4cm\mu^2)\frac{d}{dx}\sum_{i=1}^{3} d_i s_i c_i + 2\sigma^2 Bm\mu k\frac{d}{dy}\sum_{i=1}^{3} d_i s_i c_i$$

$$+4B^2 b m^2 \mu^2 \left(\sum_{i=1}^{3} d_i s_i c_i\right)^2 - 2B^2 b m\mu \sum_{i=1}^{3} d_i^2 \frac{d}{dx}\sum_{i=1}^{3} d_i s_i c_i \tag{32}$$

$$+24Bcm\mu^3 \frac{d}{dx}\sum_{i=1}^{3} d_i^3 s_i c_i = 0.$$

The last three terms can be rewritten as

$$-2B^2 b m\mu\left(-2m\mu\left(\sum_{i=1}^{3} d_i s_i c_i\right)^2 + \sum_{i=1}^{3} d_i^2 \frac{d}{dx}\sum_{i=1}^{3} d_i s_i c_i - 12\frac{c\mu^2}{bB}\frac{d}{dx}\sum_{i=1}^{3} d_i^3 s_i c_i\right), \tag{33}$$

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whereas evaluation of $\frac{d}{dx}\left(\sum_{i=1}^{3}d_i^2\sum_{j\neq i}d_js_jc_j\right)$ yields

$$
\begin{aligned}
\frac{d}{dx}\left(\sum_{i=1}^{3}d_i^2\sum_{j\neq i}d_js_jc_j\right) &= -2m\mu\left(\sum_{i=1}^{3}d_is_ic_i\right)^2 + \sum_{i=1}^{3}d_i^2\frac{d}{dx}\sum_{i=1}^{3}d_is_ic_i \\
&\quad +2m\mu\sum_{i=1}^{3}d_i^2s_i^2c_i^2 - \sum_{i=1}^{3}\left(d_i^2\frac{d}{dx}d_is_ic_i\right) \\
&= -2m\mu\left(\sum_{i=1}^{3}d_is_ic_i\right)^2 + \sum_{i=1}^{3}d_i^2\frac{d}{dx}\sum_{i=1}^{3}d_is_ic_i - \frac{d}{dx}\sum_{i=1}^{3}d_i^3s_ic_i.
\end{aligned}
\tag{34}
$$

Because $12\frac{c\mu^2}{bB} = 1$ (in Eq. (33)) holds identically, we can use Eq. (34) and [Khare *et al.* 2002b, Eq. (11)]

$$
\frac{d}{dx}\left(\sum_{i=1}^{3}d_i^2\sum_{j\neq i}d_js_jc_j\right) = A(3,m)\frac{d}{dx}\sum_{i=1}^{3}d_is_ic_i,
\tag{35}
$$

to rewrite Eq. (32) as

$$
-2Bm\mu(v_3+8c\mu^2-4cm\mu^2+BbA(3,m))\frac{d}{dx}\sum_{i=1}^{3}d_is_ic_i+2\sigma^2Bm\mu k\frac{d}{dy}\sum_{i=1}^{3}d_is_ic_i = 0.
\tag{36}
$$

Using $\frac{d}{dy}\sum_{i=1}^{3}d_is_ic_i = k\frac{d}{dx}\sum_{i=1}^{3}d_is_ic_i$ this equation reads

$$
-2Bm\mu(v_3 + 8c\mu^2 - 4cm\mu^2 - \sigma^2k^2 + BbA(3,m))\frac{d}{dx}\sum_{i=1}^{3}d_is_ic_i = 0.
\tag{37}
$$

Thus, the speed v_3 in the superposition solution (31) (of a particular case) of Eq. (17) is given by

$$
v_3 = 4cm\mu^2 + \sigma^2k^2 - 8c\mu^2 - BbA(3,m).
\tag{38}
$$

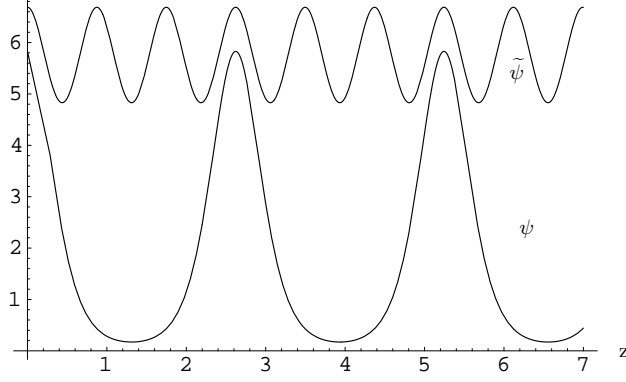The start solution and the superposition solution are shown in Fig. 2.

Figure 2: The start solution $\psi(z)$ (cf. first solution of Eq. (23)) and the super-position solution $\widetilde{\psi}(z)$ (cf. Eq. (31)) for $\alpha = -1$, $\gamma = 1$, $\epsilon = -1$, $z = x + ky + vt$ and $z = x + ky + v_3 t$, respectively.

Applying an analogous procedure with the ansatz

$$
\begin{aligned}
\widetilde{\psi}(x,y,t) &= B \sum_{i=1}^{3} \mathrm{sn}^2 \left(\mu(x + ky + v_3 t) + \frac{2(i-1)K(m)}{3}, m \right), \\
B &= B_2, \mu = \mu_2, m = m_2
\end{aligned}
\tag{39}
$$

and with the ansatz

$$
\begin{aligned}
\widetilde{\psi}(x,y,t) &= B \sum_{i=1}^{3} \mathrm{cn}^2 \left(\mu(x + ky + v_3 t) + \frac{2(i-1)K(m)}{3}, m \right), \\
B &= B_3, \mu = \mu_3, m = m_3
\end{aligned}
\tag{40}
$$

we obtain superposition solutions with

$$
v_3 = 4cm\mu^2 + 4c\mu^2 + \sigma^2 k^2 + Bb\frac{A(3,m)-2}{m}
\tag{41}
$$

for solution (39) and

$$v_3 = -8cm\mu^2 + 4c\mu^2 + \sigma^2 k^2 - Bb\frac{A(3,m) - 2(1 - m)}{m} \tag{42}$$

for solution (40).

In deriving (41) and (42) we have used the relations

$$\frac{d}{dx}\left(\sum_{i=1}^{3} \mathrm{s}_i^2 \sum_{j \neq i}^{3} \mathrm{s}_j \mathrm{c}_j \mathrm{d}_j\right) = -\frac{1}{m}\left(A(3,m) - 2\right)\frac{d}{dx}\sum_{i=1}^{3} \mathrm{s}_i \mathrm{c}_i \mathrm{d}_i \tag{43}$$

and

$$\frac{d}{dx}\left(\sum_{i=1}^{3} \mathrm{c}_i^2 \sum_{j \neq i}^{3} \mathrm{s}_j \mathrm{c}_j \mathrm{d}_j\right) = \frac{1}{m}\left(A(3,m) - 2(1 - m)\right)\frac{d}{dx}\sum_{i=1}^{3} \mathrm{s}_i \mathrm{c}_i \mathrm{d}_i, \tag{44}$$

respectively, which follow from Eq. (35) and well known relations between Jacobian elliptic functions.

Comparison of the Kadomtsev-Petivashvili equation together with the ansatz considered by Cooper, Khare and Sukhatme [Cooper *et al.* 2002, Eqs. (1),(4)]with the Eqs. (17), (19) and (31) shows that, apart from an additive constant [Jaworski *et al.* 2003], our result (38) is consistent with that of Cooper, Khare and Sukhatme [Cooper *et al.* 2002, Eq. (11), $\beta = 0$] .The cases related to (41), (42) have not been considered in Ref. [Cooper *et al.* 2002].

To conclude, we note that real and bounded suitable solutions of the gmKPE only exist for four different values of q (cf. (19) - (22)), though there is no restriction for q (apart from being real) of the known elliptic solutions of the gmKPE [Schürmann *et al.* 2004a].

The second of Eqs. (24) does not lead to a superposition solution although the solution has the form (1) [6]. In this case, it seems that an appropriate identity

for Jacobian elliptic functions does not exist. Thus, the claim at the end of Ref. [Cooper *et al.* 2002] seems to strong.

4   Summary and concluding remarks

By combining the superposition principle and symmetry reduction we obtained general elliptic solutions suitable for superposition. The results were applied to the gmKPE and the NLCQSE (see A). In Ref. [Cooper *et al.* 2002] particular elliptic solutions for generating superposition solutions of the NLSE and the KPE were used. As outlined above we start from (general) suitable solutions (cf. Eqs. (15), (16), (23), (24)) to obtain superposition solutions more general than those in Ref. [Cooper *et al.* 2002]. We note that there are no restrictions in advance for the coefficients of the NLSE and the KPE. Constraints result from the condition that suitable solutions exist (cf. Eq. (13)) and from the PDC. As is obvious from the following list I there are rather many NLWEEs that exhibit suitable elliptic solutions. Thus, it seems interesting to check whether they lead to superposition solutions by applying Eqs. (15) and (16).

Table I: Elliptic solutions of various nonlinear wave and evolution equations.

Equation	Ansatz	Basic equation	Suitable for superposition
$\psi_t - \psi\psi_x - \psi_{xxt} = 0$ Benjamin-Bona- Mahony	$\psi = f(kx - ct) = f(z)$	$(f_z)^2 = -\frac{f^3}{3kc} + \frac{f^2}{k^2}$ $+4\delta f + \epsilon$	$+$
$\psi_{tt} - \psi_{xx} + 3(\psi^2)_{xx}$ $-\psi_{xxxx} = 0$	$\psi = f(kx - ct) = f(z)$	$(f_z)^2 = \frac{2f^3}{k^2} + \frac{c^2-k^2}{k^4}f^2$ $+4\delta f + \epsilon$	$+$

Boussinesq			
$\psi_t + \psi\psi_x - \psi_{xx} = 0$ Burgers	$\psi = f(kx - ct) = f(z)$	$(f_z)^2 = \frac{(2c^2 f - kcf^2 + 4k^4\delta)^2}{4k^4 c^2}$	- [7]
$\psi_{tt} - \psi_{xx} + \sin\psi$ $+\frac{1}{2}\sin\frac{\psi}{2} = 0$ Double sine-Gordon	$\psi = 4\arctan[f(kx - ct)]$ $= 4\arctan[f(z)]$	$(f_z)^2 = 3(\gamma + \frac{1}{8(c^2-k^2)})f^4$ $+6\gamma f^2 + 3\gamma + \frac{5}{(c^2-k^2)}$, $c^2 \neq k^2$	+
$\psi_t + a\psi_{xx} - b\psi$ $-c \mid \psi \mid^2 \psi = 0$ $a = a_1 + ia_2, b = b_1 + ib_2,$ $c = c_1 + ic_2$ Ginzburg-Landau	$\psi =$ $f(x)\exp(ig(x))\exp(i\lambda t),$ $g_x(x) = d\frac{f_x}{f}$	$(f_x)^2 = \frac{c_2}{3d\,a_1 + a_2(2-d^2)}f^4$ $+ \left(\frac{\lambda_\pm - b_2}{d^2 a_2 - a_2 - 2d\,a_1}\right)f^2$ [8]	+
$\psi_t + (b\psi + 1)\psi_x + \psi_{ttx} = 0$ Joseph-Egri [Hereman *et al.* 1986]	$\psi(x,t) =$ $f(kx - ct) = f(z)$	$(f_z)^2 = -\frac{b}{3c^2}f^3 + \frac{c-k}{c^2 k}f^2$ $-\frac{2C_1}{c^2 k}f - \frac{2C_2}{c^2 k}$, C_1, C_2 const.	+
$\psi_t + a\psi^2\psi_x + b\psi_x\psi_{xx} +$ $g\psi\psi_{xxx} + \psi_{xxxx} = 0$ Korteweg-de Vries	$\psi(x,t) =$ $f(x - ct) = f(z)$	$(f_z)^2 = 4\beta f^3 + 6\gamma f^2$ $+4\delta f + \epsilon$	+
$\psi_t - \psi_{xx} - 6\psi\psi_x + \psi_{xxx}$ $= 0$ Korteweg-de Vries- Burgers	$\psi = f^2(kx - ct) = f^2(z)$	$(f_z)^2 = \frac{f^2}{k^2} \cdot$ $\cdot \left(\frac{f^2}{2} \pm \frac{2f}{5\sqrt{2}} + \frac{1}{25}\right)$	- [9]
$\psi_t + a\psi\psi_x + \psi_{xxx}$ $+\psi_{xyy} + \psi_{xzz} = 0$ KdV-Zakharov- Kusnetzov [Baldwin *et al.* 2004]	$\psi(x,y,z,t) = f(\xi)$ $\xi = kx + ly + mz + \omega t$	$(f_\xi)^2 = -\frac{a}{3p^2}f^3$ $-\frac{\omega}{kp^2}f^2 - \frac{2C_1}{kp^2}$, $p^2 = k^2 + l^2 + m^2$, C_1 const.	+

$V_t = \partial^3 V + \overline{\partial}^3 V$ $+3\partial(uV) + 3\overline{\partial}(\overline{u}V),$ $\overline{\partial}u = \partial V$ Novikov-Veselov	$V(x,y,t) = \psi(z)$ $z = x + ky - vt$	$(f_z)^2 = -\frac{8a_2}{1+k^2}f^3$ $-\frac{24a_0}{1+k^2}f^2 + \frac{12F}{3(3k^2-1)}f^2$ $+4\delta f + \epsilon,$ $F = v + 3C_0 + 3kC_1;$ C_0, C_1 const.	+
$\psi_{xx} - \psi_{tt} - \sin\psi = 0$ sine-Gordon	$\psi(x,t) = 4\arctan[\frac{X(x)}{T(t)}]$	$(\frac{dX}{dx})^2 = R_1(X)$ $= \alpha X^4 + 6\gamma X^2 + \epsilon$ $(\frac{dT}{dt})^2 = R_1(T)$ $= \alpha T^4 + (6\gamma-1)T^2 - \epsilon$	+
$i\psi_x + \psi_{tt} + 2\sigma|\psi|^2\psi$ $-\mu\psi_{xt} = 0$ Wadati, Segur, Ablowitz [Wadati *et al.* 1992]	$\psi(x,t) = f(z)e^{i(rx-\lambda t)}$ $z = kx - ct$	$(f_z)^2 = -\frac{\sigma}{c(c+k\mu)}f^4$ $-\frac{k(1+\lambda\mu)^2+c\lambda(2+\lambda\mu)}{c^2\mu(c+k\mu)}f^2$ $-\frac{2C_1}{c(c+k\mu)}, C_1$ const.	+

If condition (13) can be fulfilled (e. g., by choosing an appropriate constant of integration) start solutions for superposition can be obtained by Eqs. (15), (16) (marked by a "+", otherwise by a "-").

Acknowledgements

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A Superposition solutions of the nonlinear cubic-quintic Schrödinger equation (NLCQSE)

Following the lines described in Secs. 1, 2 the NLCQSE

$$i\psi_t + \psi_{xx} - (q_1|\psi|^2 + q_2|\psi|^4)\psi = 0, \tag{45}$$

(q_1, q_2 real constants) can be solved by applying the transformation

$$\psi(x,t) = f(z)\exp[i(\lambda t + r(z))], z = x - ct. \tag{46}$$

Separating real and imaginary parts, we obtain

$$q_1 f(z)^3 + q_2 f(z)^5 - f''(z) + f(z)(\lambda - cr'(z) + r'(z)^2) = 0, \tag{47}$$

$$f'(z)(c - 2r'(z)) - f(z)r''(z) = 0, \tag{48}$$

where the prime denotes differentiation with respect to z.

Equation (48) can be integrated to yield

$$r'(z) = \frac{c}{2} + \frac{C_1}{f(z)^2}, \tag{49}$$

with C_1 a constant of integration.

Inserting $r'(z)$ into Eq. (47) and integrating the resulting expression leads to an equation where $h = f^2$ can be introduced. Thus, we find a basic equation $R(h)$ (cf. Eq. (3), $f \to h$) with the following coefficients:

$$\alpha = \frac{4}{3}q_2,\ \beta = \frac{1}{2}q_1,\ \gamma = \frac{4\lambda - c^2}{6},\ \delta = 2\,C_2, \epsilon = -4\,C_1^2, \tag{50}$$

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where C_2 is a constant of integration.

If $q_2 = 0$ and $C_1 = 0$ all physical solutions suitable for superposition are represented by Eqs. (15) $(f \to h)$. The superposition solutions for $p = 3$ are given by (cf. Eqs. (2), (46))

$$\widetilde{\psi}(x,t) = a \sum_{i=1}^{3} \mathrm{cn}\left[\mu(x - v_3 t) + \frac{4(i-1)K(m)}{3}, m\right] \exp\left\{i\left[\lambda t + (x - v_3 t)\frac{v_3}{2}\right]\right\},$$
$$v_3^2 = 4(\lambda - \mu^2(2mX(3,m) + (2m - 1))), \tag{51}$$

$$\widetilde{\psi}(x,t) = a \sum_{i=1}^{3} \mathrm{dn}\left[\mu(x - v_3 t) + \frac{2(i-1)K(m)}{3}, m\right] \exp\left\{i\left[\lambda t + (x - v_3 t)\frac{v_3}{2}\right]\right\},$$
$$v_3^2 = 4(\lambda + \mu^2(m - 2) - aW(3,m)), \tag{52}$$

$$\widetilde{\psi}(x,t) = a \sum_{i=1}^{3} \mathrm{sn}\left[\mu(x - v_3 t) + \frac{4(i-1)K(m)}{3}, m\right] \exp\left\{i\left[\lambda t + (x - v_3 t)\frac{v_3}{2}\right]\right\},$$
$$v_3^2 = 4(\lambda + \mu^2(m + 1) + 2ma\mu^2 V(3,m)). \tag{53}$$

To evaluate the speed v_3 we have used in Ref. [Cooper *et al.* 2002] the Eqs. (8), (70), (72), Eqs. (8), (66), (68) and Eqs. (8), (57), (59), respectively.

It should be mentioned that the start solutions (15) suitable for superposition are consistent with those of Cooper, Khare and Sukhatme [Cooper *et al.* 2002]. Nevertheless, the speed v_3 according to Eqs. (51), (52), (53) is not identical with v_3 according to Eqs. (33), (28), (45) in Ref. [Cooper *et al.* 2002]. Thus, the superposition solutions are not determined uniquely. Different identities between Jacobian elliptic functions used lead to (in general) different superposition solutions. Applying the procedure outlined in Sec. 2 if $q_2 \neq 0$ ($\alpha \neq 0$), $\beta = \delta = 0$, ϵ arbitrary, PDC implies either $q_2 = 0$ ($\alpha = 0$) or $C_1^2 = 0$ ($\epsilon = 0$). The choice $q_2 = 0$ (in addition to $q_1 = 0$ ($\beta = 0$)) is not of interest, because it leads to a linear equation

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(45). For $C_1^2 = 0$ we obtain solitary traveling-waves. Thus, since $\psi(x,t)$ is not periodic, superposition solutions are not possible in this case.

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[1] The general solution of Eq. (3) reads [Weierstrass 1915, Whittaker *et al.* 1927]

$$f(z) = f_0 + \frac{\sqrt{R(f_0)}\frac{d\wp(z;g_2,g_3)}{dz} + \frac{1}{2}R'(f_0)[\wp(z;g_2,g_3) - \frac{1}{24}R''(f_0)] + \frac{1}{24}R(f_0)R'''(f_0)}{2[\wp(z;g_2,g_3) - \frac{1}{24}R''(f_0)]^2 - \frac{1}{48}R(f_0)R''''(f_0)}.$$

[2] We assumed $\triangle > 0$. If $\triangle < 0$, substitution of $\wp(z)$ according to Ref. [Abramowitz *et al.* 1972], 18.9.11 does not lead to an expression of form (1). Furthermore, as will be seen below, we obtain a polynomial $R(f)$ or $R(h)$, $h = f^2$, of third degree for generating new solutions by linear superposition. But the PDC is not satisfied for a third-degree polynomial with $\triangle < 0$, because two of the three roots are complex conjugate.

[3] Since we are interested in physical periodic solutions we can always assume that a simple root exists.

[4] If $\beta = 0$ holds γ must be negative otherwise Eq. (13) and PDC are not fulfilled. For f_0 to be a simple root $\delta^2 - \frac{3}{2}\epsilon\gamma$ must be positive. If $\beta \neq 0$ the discriminant $\triangle$ does not vanish, so that $f_0 = \frac{2e_3 - \gamma}{2\beta}$ is a simple root of $R(f)$.

[5] If parameters according to Eqs. (21) are inserted into Eq. (16) one obtains B, μ, m so that $6Bcm\mu^3 - m\mu bB^3$ in Eq. (28) vanishes identically.

[6] An ansatz $\widetilde{\psi} = $ cn or $\widetilde{\psi} = $ sn (cf. eq. (25)) leads to equations which have the form (26). Because there is no relation $c_1c_2 = $ const. and $s_1s_2 = $ const., respectively, there is no possibility to replace the appearing sums $\sum c_i \left(\sum d_i s_i\right)^2$, $\left(\sum c_i\right)^2 \frac{d}{dx} \sum d_i s_i$ and $\sum s_i \left(\sum c_i d_i\right)^2$, $\left(\sum s_i\right)^2 \frac{d}{dx} \sum c_i d_i$, respectively. Up to our knowledge there is no appropriate relation involving Jacobian elliptic functions that would simplify the equations similar to (26), so that the speed v_2 for which $\widetilde{\psi}$ is a solution of eq. (17) cannot be determined.

[7] As outlined above start solutions for linear superposition can be obtained if $\alpha = 0$ or $\beta = \delta = 0$. In the case of the Burgers equation these conditions lead to $k \to \infty$ or $c = 0$ for a traveling wave ansatz $\psi = f(kx - ct)$.

[8] d only depends on the coefficients of the Ginzburg-Landau equation a, b, c.

[9] As outlined above start solutions for linear superposition can be obtained if $\alpha = 0$ or $\beta = \delta = 0$. In the KdV-Burgers equation the only parameter which can be varied in the basic equation is k. If $k \to \infty$ then $\alpha \to 0$, but also $(f_z)^2 \to 0$ so that $f \equiv \mathrm{const.}$.